Permutation packing and Flag Algebras

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- H, F, G graphs/permutations
- G is F-free
- $|H| = k$, $|G| = n$

$$\mathbf{p(H, G)} = \frac{\#\text{ induced copies of } H \text{ in } G}{\binom{n}{k}}$$

Maximize the density of H in an F-free G.

$\mathcal{F}$ a family of forbidden permutations/graphs

$$p(H, n) = \max_{\substack{G \text{ is } \mathcal{F}\text{-free} \\ |G|=n}} p(H, G)$$

Packing density

$$p(H) = \lim_{n \to \infty} p(H, n)$$

Turán density

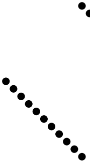
$$p(H, \mathcal{F}) = \lim_{n \to \infty} p(H, \mathcal{F}, n)$$

Example I

Permutations

Maximize density of 12 in a 123-free P.

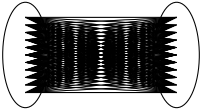
Answer: $\sim 1/2$.



Graphs

Maximize the density of  in a -free G.

Answer: $\sim 1/2$.



Example II

Permutations

Minimize the density of monotone subsequences of length 4.
Answer: $\binom{\lfloor n/3 \rfloor}{4}\binom{\lfloor n/3+1 \rfloor}{4}\binom{\lfloor n/3+2 \rfloor}{4}$ via Flag algebras [BHL+15].

Graphs

Minimize the density of  $+$ **.**
Notoriously hard. Minimizer is NOT $\mathbb{G}(n, 1/2)$ — can do better.
Still open!

Example III

Permutations

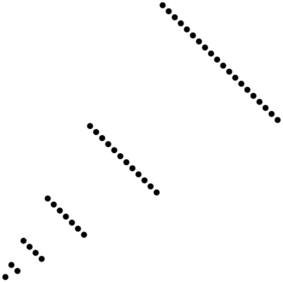
Maximize the density of 132 permutation.

Answer: $2\sqrt{3} - 3$ (Galvin-Kleitmann, Stromquist).

Digraphs

Maximize the density of **.**

Answer: $2\sqrt{3} - 3$ ([FRV13]).



Timeline

1992	Wilf @ SIAM	*please look at packing densities*
1992	Galvin	packing densities exist
1993	Galvin-Kleitmann, Stromquist	132 packing density
1998	Price	PhD thesis, layered patterns
2002	Albert et al.	packing densities of layered patterns (+ LBs for 1342, 2413)
2002	Hästö	packing density of *other* layered permutations
2006	Barton	packing densities of patterns
2008	Presutti	lower bounds for packing non-layered patterns (weighs AAHHS templates)
2010	Presutti-Stromquist	packing rates of measures, LB for 2413
2015	Balogh et al.	Minimum number of monotone 4-point sequences

In particular...

Structural result (Stromquist, [AAH$^+$02])

Layered (on top) permutations pack best into layered (on top) permutations.

Must mention:

- Price's algorithm
- Generalization to patterns
- Number of layers – bdd or not
- Presutti-Stromquist permutation limits
- Packing densities of linear combinations with coeffs in $\mathbb{N}$

Non-layered (before FA):

- $0.1047\ldots \leq p($$) \leq 2/9$
- $0.1965\ldots \leq p($$) \leq 2/9$

Small permutations (overview)

3-point packing densities

Done.

4-point packing densities

S	lower bound	ref LB	upper bound	ref UB
1234	1	trivial	1	trivial
1432	0.42357...	[Pri97]	0.42357...	[Pri97]
2143	3/8	trivial	3/8	[Pri97]
1243	3/8	trivial	3/8	[AAH$^+$02]
1324	$\approx$ 0.244...	[Pri97]	$\approx$ 0.244...	[Pri97]
1342	0.1965...	[Bat]	2/9	[AAH$^+$02]
2413	0.104724...	[PS10]	2/9	[AAH$^+$02]

Strategy

First bound

Maximize density of K_2 in a K_3-free graph.

$$p(K_2, G) = p(K_2, \overline{K_3})p(\overline{K_3}, G) + p(K_2, K_2 \cup K_1)p(K_2 \cup K_1, G) + p(K_2, P_3)p(P_3, G)$$
$$\leq \max\{p(K_2, \overline{K_3}), p(K_2, K_2 \cup K_1), p(K_2, P_3)\} = 2/3$$

Redistribute weight

$$p(K_2, G) = \sum_{H \in \mathcal{G}_k} p(K_2, H)p(H, G) + \underbrace{\sum_{H \in \mathcal{G}_k} c_H p(H, G)}_{\geq 0}$$
$$\leq \max_{H \in \mathcal{G}_k} p(K_2, H) + c_H \quad // \text{ winning if the right } c_H \text{ negative}$$

Magic (flag algebras)

- $\mathcal{G} = \bigcup_{n \geq 0} \mathcal{G}_n$ set of all graphs.
- $\mathcal{A} = \mathbb{R}\mathcal{G}/\mathcal{K}$, where $\mathcal{K}$ contains $H - \sum_{H' \in \mathcal{G}_k} p(H, H')H'$.
- $H_1 \cdot H_2 := \sum_H p(H_1, H_2; H)H$ (now $\mathcal{A}$ is an algebra).
- $(G_n)_n$ convergent if $(p(H, G_n))_n$ converges for every H.
- Every $(G_n)_n$ has convergent subsequence (Tychonoff).
- Associate convergent $(G_n)_n$ with $\phi \in Hom^+(\mathcal{A}, \mathbb{R})$.
- Averaging: $H^\sigma \in \mathcal{A}^\sigma$, $[\![H^\sigma]\!]_\sigma = p_H^\sigma H$.
- For every $\phi \in Hom^+(\mathcal{A}, \mathbb{R})$ and every $h^\sigma \in \mathcal{A}^\sigma$,

$$\phi\left([\![h^\sigma]\!]_\sigma\right) \geq 0.$$

- Semi-definite programming, additional info comes from:

$$\phi\left(\left[\!\!\left[\mathbf{x}^T Q \mathbf{x}\right]\!\!\right]\right) \geq 0, \quad Q \succeq 0,$$

  where $\mathbf{x}_i \in \mathcal{A}^\sigma$.

Permpack package

Sage package automating flag algebras for permutations.

```
https://github.com/jsliacan/permpack.git
```

Ingredients

- Razborov's Flag Algebras [Raz07]
- Sage-7.2 (SageMath.org)
- CSDP (COIN|OR project)

What to expect

- Maximize packing density for $\pi = \sum_{i=1}^{k} a_i P_i$, some k.
- Compute Turán density for finite $\mathcal{F}$ and π
- Handle inequality constraints, i.e.  $\geq 1/3$

Permpack example I

Listing 1: Packing 132

```
p = PermProblem(3, density_pattern="132")
p.solve_sdp(solver="csdp")
```

Listing 2: Output

```
...
Success: SDP solved
Primal objective value: -4.6410162e-01
Dual objective value: -4.6410162e-01
Relative primal infeasibility: 5.90e-14
Relative dual infeasibility: 1.67e-10
Real Relative Gap: 3.68e-10
XZ Relative Gap: 6.14e-10
```

Permpack example II

Listing 3: Packing 123 subject to 132 $\geq 1/3$.

```
p = PermProblem(3, density_pattern="123")
p.add_assumption([("132",1)], 1/3)
p.solve_sdp(solver="csdp")
```

Listing 4: Output

```
...
Success: SDP solved
Primal objective value: -5.9221808e-01
Dual objective value: -5.9221809e-01
Relative primal infeasibility: 3.14e-15
Relative dual infeasibility: 2.48e-09
Real Relative Gap: -9.65e-10
XZ Relative Gap: 4.52e-09
```

After Flag Algebras

S	lower bound	ref LB	upper bound	ref UB
1234	1	trivial	1	trivial
1432	α	[Pri97]	α	[Pri97]
2143	3/8	trivial	3/8	[Pri97]
1243	3/8	trivial	3/8	[AAH+02]
1324	0.244054321	construction Γ	0.244054549	Flagmatic 2.0
1342	β	[Bat]	0.1988373...	[BHL+15]
2413	≈ 0.104724	[PS10]	0.1047805...	[BHL+15]

All known upper bounds can be re-proved via FA.

The curious case of 1342

$\Gamma =$ 

Batkeyev, [BHL+15]

$$0.1965... = p(1342, \Gamma) \leq 1342 \leq 0.1988....$$

Theorem

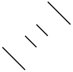
$$\text{If } 2431 = 0, \text{ then } 1342 \leq 0.19658...$$

Various non-layered densities

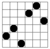
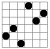
All bounds are matched from below by constructions on the left.

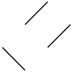
 $\leq 0.16039\ldots$

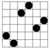
 $\leq 0.153649\ldots$



 $\leq 0.16515\ldots$ [Häs02]: $[k, 1, k]$, $k \geq 3$

 $=$  $= (2\sqrt{3} - 3)^2 \frac{6!}{48^2} \leq 0.0673094$



 $\leq 0.123456\ldots$

Things to think about

Do theorems about layered permutations extend to sums of sum-indecomposable blocks?

Is there a -maximizer that does not contain  with positive density?

Which sub-permutation densities force G to be pseudo-random?

...

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Electron. J. Combin, 9(1), 2002.

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Combinatorics, Probability and Computing, 24(04):658–679, 2015.

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The Journal of Symbolic Logic, 72(04):1239–1282, 2007.