Well-behaved $1 \times n$ permutation grid classes

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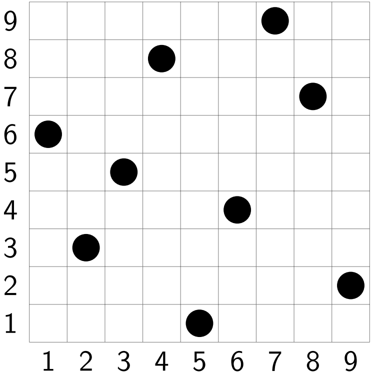
View permutations as drawings

635814972 $\longleftrightarrow$ 

Enumerating permutation classes

Class

Collection of permutations $\mathcal{C}$ together with a *size (length) function* $|\cdot|$ on elements of $\mathcal{C}$.

Enumeration

Determining the number of permutations of each length in $\mathcal{C}$ yields an *enumeration sequence*.

Example

Class $\mathrm{Av}(123, 321)$ is enumerated by the sequence $1, 1, 2, 4, 4, 0, 0, 0 \ldots$ (zeros follow from Erdős-Szekeres).

Storing enumeration sequences

Generating functions

Infinitely long objects need a suitable data structure to be stored in finite space – *generating functions (gfs)* FTW!

Example

Store $1, 1, 1, 1, \ldots$ as coefficients of $1/(1 - z) = \sum_{k=0}^{\infty} z^k$.

Not all gfs are made equal:

$$\underbrace{\text{rational}}_{\textit{easy}} \subset \underbrace{\text{algebraic}}_{\textit{nice}} \subset \underbrace{\text{D-finite} \subset \text{non D-finite} \subset \text{power series}}_{\textit{other}}$$

Grid classes

Definition

Permutation *grid class* is a permutation class. It consists of permutations that can be chopped up by vertical and horizontal lines into sub-permutations belonging to designated classes.

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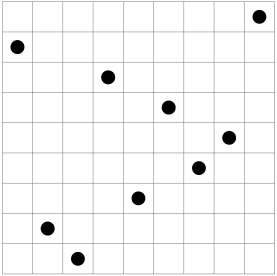

belongs to $\begin{bmatrix} \mathrm{Av}(12) & \mathrm{Av}(21) \\ \mathrm{Av}(12) & \mathrm{Av}(21) \end{bmatrix}$.

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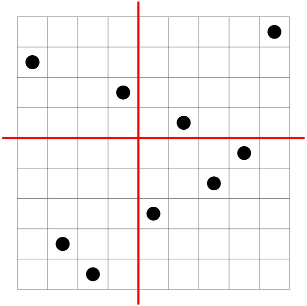
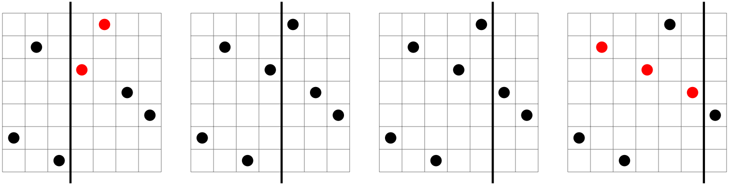


belongs to $\begin{bmatrix} \mathrm{Av}(12) & \mathrm{Av}(21) \\ \mathrm{Av}(12) & \mathrm{Av}(21) \end{bmatrix}$.

Example: non-unique gridding

2615743 is in $\begin{array}{|c|c|}\hline \mathrm{Av}(321) & \mathrm{Av}(12) \\ \hline \end{array}$ as witnessed by the middle two partitions.



No! Yes! Yes! No!

Ideally, we'd like to enumerate these

$\mathcal{C}_{11}$	$\mathcal{C}_{12}$	$\mathcal{C}_{13}$				$\mathcal{C}_{1m}$
$\mathcal{C}_{21}$	$\mathcal{C}_{22}$	$\mathcal{C}_{23}$				$\mathcal{C}_{2m}$
$\mathcal{C}_{31}$	$\mathcal{C}_{32}$	$\mathcal{C}_{33}$			$\cdots$	$\mathcal{C}_{3m}$
		$\vdots$			$\ddots$	
$\mathcal{C}_{n1}$	$\mathcal{C}_{n2}$	$\mathcal{C}_{n3}$				$\mathcal{C}_{nm}$

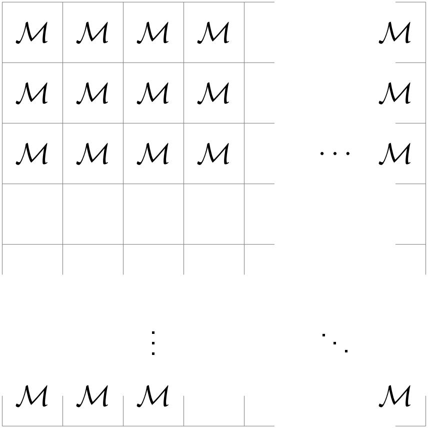
Even if $\mathcal{C}_{ij}$ are permutation classes that we CAN enumerate

. . . or this

$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$			$\mathcal{M}$
$\mathcal{M}$	$\mathcal{M}$	$\mathcal{C}$	$\mathcal{M}$			$\mathcal{M}$
$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$		$\cdots$	$\mathcal{M}$
		$\vdots$			$\ddots$	
$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$				$\mathcal{M}$

$\mathcal{M}$ monotone classes, $\mathcal{C}$ non-monotone class

. . . actually, not even this



$\mathcal{M}$ monotone classes

But! we know their growth rates = (spectral radius)^2 of the row-column graph [Bev15a].

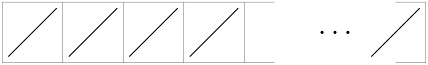
. . . also . . .

these have rational generating functions [AAB⁺13]

$$
\mathrm{Geom}\begin{pmatrix}
\mathcal{M} & \mathcal{M} & \mathcal{M} & \mathcal{M} & & & \mathcal{M} \\
\mathcal{M} & \mathcal{M} & \mathcal{M} & \mathcal{M} & & & \mathcal{M} \\
\mathcal{M} & \mathcal{M} & \mathcal{M} & \mathcal{M} & & \cdots & \mathcal{M} \\
 & & & & & & \\
 & & \vdots & & & \ddots & \\
\mathcal{M} & \mathcal{M} & \mathcal{M} & & & & \mathcal{M}
\end{pmatrix}
$$

. . . and . . .

generating functions conjectured for monotone increasing strips [Bev15b]



New: monotonely padded context-free classes are nice

$\mathcal{M}_1$		$\cdots$		$\mathcal{M}_k$	$\mathcal{C}$	$\mathcal{M}_{k+1}$	$\cdots$	$\mathcal{M}_{k+l}$

Theorem (approx.)

If $\mathcal{C}$ is a context-free [regular] permutation class and $\mathcal{M}_i$ are monotone permutation classes, then $\mathcal{M}_1|\dots|\mathcal{M}_k|\mathcal{C}|\mathcal{M}_{k+1}|\dots|\mathcal{M}_{k+\ell}$ is a context-free [regular] class (and admits an algebraic [rational] generating function).

Context-free class

Definition

A class $\mathcal{C}$ is *context-free* if it can be characterised by the following system of equations, where $\mathcal{C} = \mathcal{S}_1$.

$$\begin{cases} \mathcal{S}_1 & = f_1(\mathcal{Z}, \mathcal{S}_1, \dots, \mathcal{S}_r) \\ & \vdots \\ \mathcal{S}_r & = f_r(\mathcal{Z}, \mathcal{S}_1, \dots, \mathcal{S}_r) \end{cases}$$

where f_i are constructors only involving $+$, $\times$, and $\mathcal{E} = \emptyset$.

Regular class

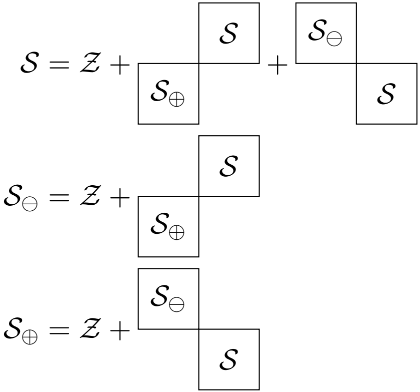
Definition

A class $\mathcal{C}$ is *regular* if it can be characterised by the following system of equations, where $\mathcal{C} = \mathcal{S}_1$.

$$
\begin{cases}
\mathcal{S}_1 & = f_1(\mathcal{Z}, S_1, \ldots, \mathcal{S}_r) \\
& \vdots \\
\mathcal{S}_r & = f_r(\mathcal{Z}, S_1, \ldots, \mathcal{S}_r)
\end{cases}
$$

where f_i are constructors only involving atoms and operations $+$, $\times$, and $\textsc{Seq}[\cdot]$.

Example of context-free class: Separables



$$\begin{aligned} \mathcal{S} &= \mathcal{Z} + \mathcal{S}_\oplus \mathcal{S} + \mathcal{S}\mathcal{S}_\ominus \\ \mathcal{S}_\ominus &= \mathcal{Z} + \mathcal{S}_\oplus \mathcal{S} \\ \mathcal{S}_\oplus &= \mathcal{Z} + \mathcal{S}\mathcal{S}_\ominus, \end{aligned}$$

Context-free classes are nice

Theorem (Chomsky-Schützenberger)

A context-free combinatorial class $\mathcal{C}$ admits an algebraic generating function.

Many things are context-free, e.g.

$$\text{finitely many simples} \implies \text{context-free}$$

Before anything else...

Ensure unique gridding

Associate RHS sequences with LHS points

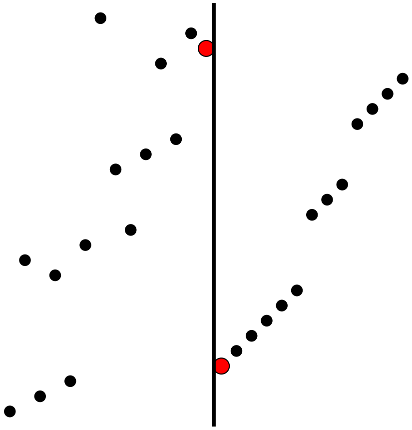
Record Cartesian product bottom to top

Leftmost gridlines

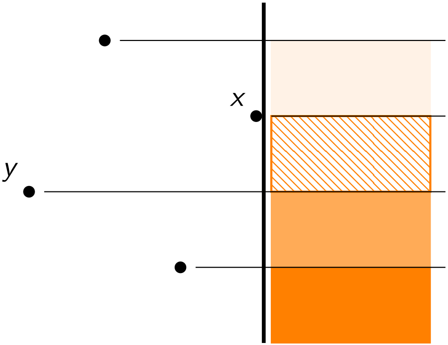
Griddable $\to$ gridded

Convention:
Let π be a permutation from $\mathcal{C}_1|\mathcal{C}_2$. The gridline in π is chosen to be the left-most possible. I.e. if it was any further left, the sub-permutation to the right of it would not belong to the designated class $\mathcal{C}_2$.

Leftmost gridlines: example $\mathcal{C}|\mathrm{Av}(21)$

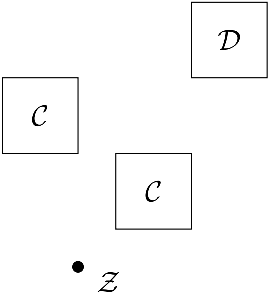


Gaps associated with points



The *gap associated with* x is the space on the RHS below x and above the next point below it on the LHS.

Bottom-to-top Cartesian products



An example of a class which would correspond to the term $\mathcal{Z}\mathcal{C}\mathcal{C}\mathcal{D}$ in a bottom-to-top combinatorial specification.

Strategy

1. Append monotone classes one at a time.
2. Break down "appending" into simpler steps.
3. Set up a standalone operator for each step.
4. Prove "context-freeness invariant" for each operator.
5. Coordinate how operators act alongside each other.
6. Establish an "appending invariant".
7. Put everything together.

Appending invariant

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Appending invariant

if $\mathcal{C}$ *is context-free, then* $\mathcal{C}|\mathrm{Av}(21)$ *is context-free*

depends on items 2.–5. We'll observe it holds once 2.–5. are done.

Appending step-by-step

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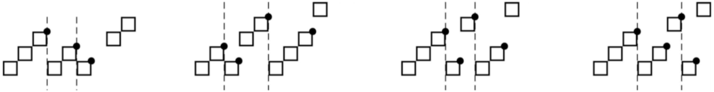
What we want to do: example

Enumerate $\mathrm{Av}(21|21|21)$. Append cells from left to right.

1. Start with a single increasing sequence on the LHS.

2. Now append stuff on the RHS.

3. Finally, append the third cell.



Tracking the rightmost point

The rightmost point of $\mathcal{C}$ is critical. So pick the combinatorial specification of $\mathcal{C}$ that tracks the rightmost point.

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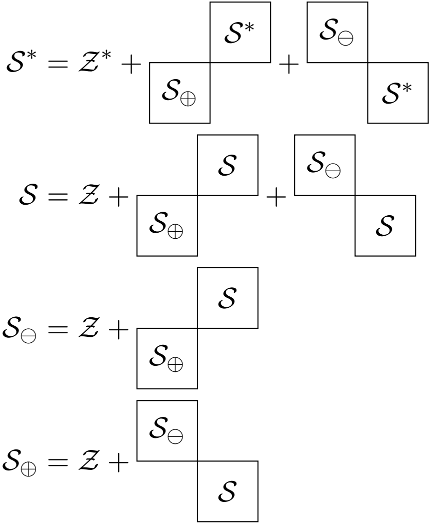

$$
\begin{aligned}
\mathcal{S}^* &= \mathcal{Z}^* + \mathcal{S}_\oplus \mathcal{S}^* + \mathcal{S}^* \mathcal{S}_\ominus \\
\mathcal{S} &= \mathcal{Z} + \mathcal{S}_\oplus \mathcal{S} + \mathcal{S} \mathcal{S}_\ominus \\
\mathcal{S}_\ominus &= \mathcal{Z} + \mathcal{S}_\oplus \mathcal{S} \\
\mathcal{S}_\oplus &= \mathcal{Z} + \mathcal{S} \mathcal{S}_\ominus .
\end{aligned}
$$

Designing operators

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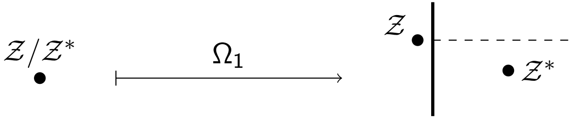
Operators 1

Consider Ω_1, an operator that appends a single point on the right of a term $\mathcal{T}_m = X_1 \ldots X_m$ (bottom to top). Rightmost point is in kth block.

$$\Omega_1(\mathcal{Z}) = \mathcal{Z}^* \mathcal{Z}$$

$$\Omega_1(\mathcal{Z}^*) = \mathcal{Z}^* \mathcal{Z}$$

$$\Omega_1(\mathcal{T}_m) = \begin{cases} \Omega_1(X_1^*)\Omega_0(X_2 \cdots X_m) & \text{if } k = 1 \\ \Omega_1(X_1)\Omega_0(X_2 \cdots X_m) + \Omega_0(X_1)\Omega_1(X_2 \cdots X_m), & \text{if } k > 1. \end{cases}$$



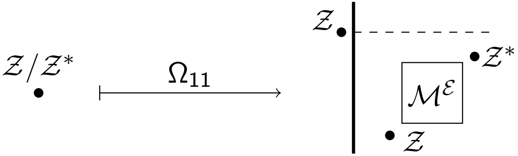
Operator 2

Ω_{11} is the most involved operator – placing a sequence on the RHS with designated bottom and top point. Rightmost point in kth block.

$$\Omega_{11}(\mathcal{Z}) = \mathcal{Z}(\mathcal{M}+\mathcal{E})\mathcal{Z}^*\mathcal{Z}$$

$$\Omega_{11}(\mathcal{Z}^*) = \mathcal{Z}(\mathcal{M}+\mathcal{E})\mathcal{Z}^*\mathcal{Z}$$

$$\Omega_{11}(\mathcal{T}_h) = \begin{cases} \left.\begin{array}{l} \Omega_{11}(X_1^*)\Omega_0(X_2\cdots X_m)+ \\ \quad +\Omega_{10}(X_1^*)\Omega_{01}(X_2\cdots X_m) \end{array}\right\} & \text{if } k=1 \\ \left.\begin{array}{l} \Omega_{11}(X_1)\Omega_0(X_2\cdots X_m)+ \\ \quad +\Omega_{10}(X_1)\Omega_{01}(X_2\cdots X_m)+ \\ \quad +\Omega_0(X_1)\Omega_{11}(X_2\cdots X_m) \end{array}\right\} & \text{if } k>1. \end{cases}$$

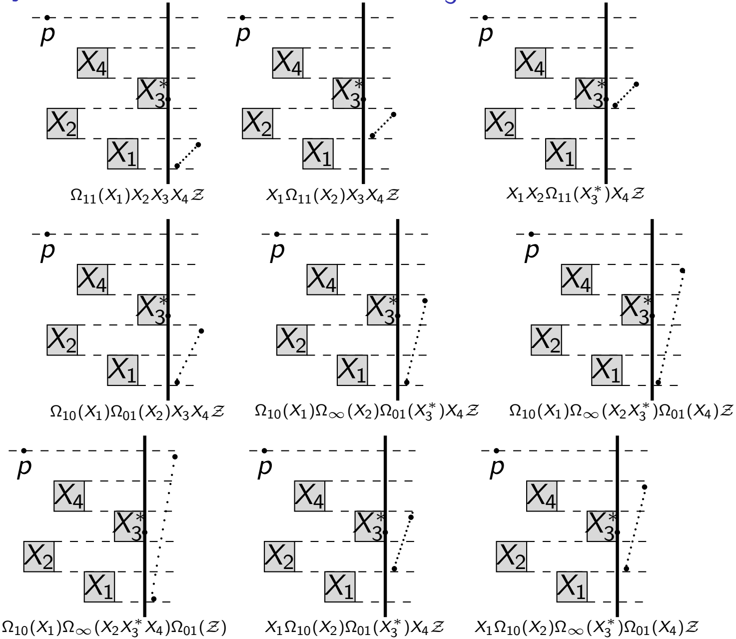


All operators

We need the following information captured when appending sequences on the RHS.

- Ω_0: Nothing appended on the RHS.
- Ω_1: Single point appended on the RHS (leftmost & rightmost coincide)
- Ω_∞: Possibly empty sequence by itself.
- Ω_{10}: Point followed by a (possibly empty) sequence above.
- Ω_{01}: Point preceded by a (possibly empty) sequence below.
- Ω_{11}: Point followed by a (possibly empty) sequence followed by another point.

Apply Ω_{11} to a class $\mathcal{C} = X_1X_2X_3^*X_4$



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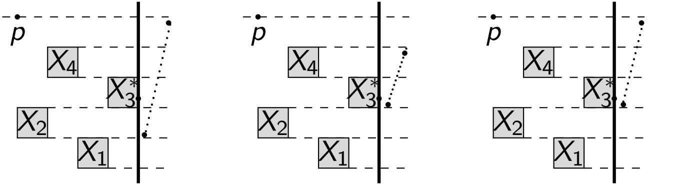

$X_1 \Omega_{10}(X_2)\Omega_\infty(X_3^* X_4)\Omega_{01}(\mathcal{Z})$ $\quad$ $X_1 X_2 \Omega_{10}(X_3^*)\Omega_{01}(X_4)\mathcal{Z}$ $\quad$ $X_1 X_2 \Omega_{10}(X_3^*)\Omega_\infty(X_4)\Omega_{01}(\mathcal{Z})$

Correctness

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- Each operator only depends (via $+, \times$) on other operators and classes. So the specification is finite & closed.
- Action on atoms is trivial.
- Hence, appending invariant holds: if $\mathcal{C}$ context-free, then $\mathcal{C}|\mathrm{Av}(21)$ context-free.

Final touches

1. Append monotone classes one at a time.
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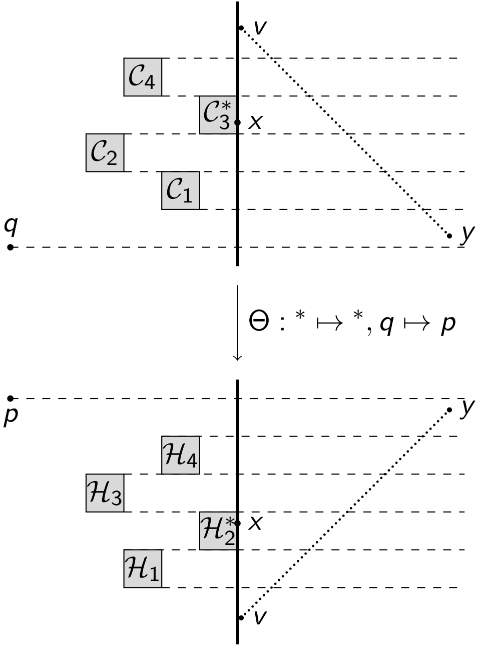
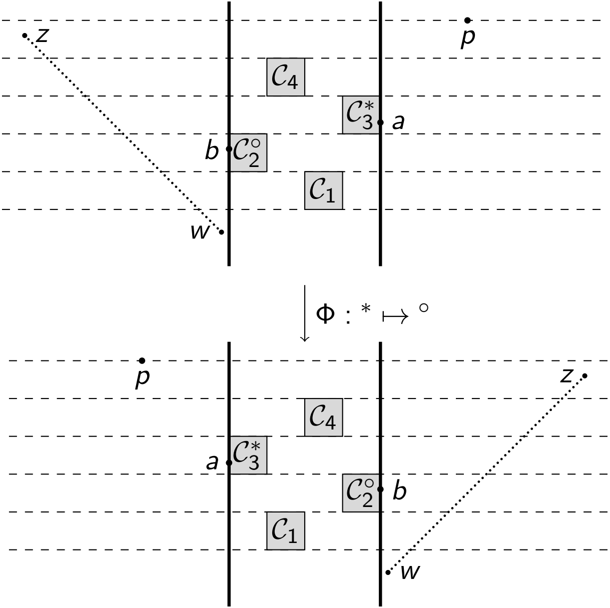
Symmetry operators

Appending a monotone decreasing class



Appending on the left



Combining everything...

! This is incorrect, but close enough !

$$\mathcal{C}|\mathrm{Av}(21) = \mathcal{E} + \mathcal{M} + \Omega_1(\mathcal{C}^*) + \Omega_{11}(\mathcal{C}^*)$$

- For iterated juxtapositions, we set $\mathcal{C} := \mathcal{C}|\mathrm{Av}(21)$, and continue as before (possibly first pre-processing with symmetry operators).
- Start with *regular* $\mathcal{C}$, end up with *regular* $1 \times n$ class.
- Enumerated exactly:
  - $\mathcal{S}|\mathrm{Av}(21)$
  - $\mathrm{Av}(132)|\mathrm{Av}(21)$
  - $\mathrm{Av}(21)|\mathrm{Av}(21)|\mathrm{Av}(21)$

Things to notice

- algorithmic approach $\rightarrow$ can be automated
- it's constructive: can enumerate (provide g.f. for) every such $1 \times n$ grid class
- D-finite?
- $n \times m$ acyclic grid classes?
- etc.

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On the growth of permutation classes.
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